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A NEW MODEL FOR THE DYNAMICS OF WETTING FILMS

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The slow motion of wetting films of a viscous fluid in thin capillaries is considered. It is known that the gas–fluid interface (a meniscus) in a capillary can form a dynamic boundary angle with a solid wall [1, 2]. For small angles, a stationary precursive film (p-film) can form, which moves under the action of van der Waals forces. Its characteristic maximum thickness depends on the velocity [1].

A mathematical model of the nonstationary dynamics of wetting films [3] includes boundary conditions at the moving boundary with a "thick" film and at the wetting line. A new model describes slow wetting without a boundary angle. The nonlinear boundary-value problem corresponds to nonsteady flow of a film with two boundary layers. Criteria of realization of the ultimate modes of the dynamics of a p-film are found. It is also shown that in flow with a boundary angle, the length of the p-film is much shorter than the capillary radius. Some significant effects of the dynamics of wetting films can be described only within the framework of the new model. The dynamics of a p-film during fluid rise in a thin capillary can be realized largely by this model.

A general analytical solution of the nonlinear self-similar wetting problem in the flat case is obtained. Previously, the main asymptotes and an analytical solution of the axisymmetric problem [3] were found. It is established that the problems of wetting of a dry surface and of a surface covered with a superthin film are similar, as are the problems of spreading of wetting films in the cases of a stationary meniscus, a semi-infinite film, and discontinuity decay for thickness. The limits of validity for the resulting self-similar solutions as intermediary asymptotes for $t \to \infty$ are found in which we ignore the influence of gravity.

1. Equilibrium Interface and the Dynamics of a Wetting Film. The spreading motion of a thin wetting film of a viscous fluid over a flat solid surface in the presence of gravity **g** and van der Waals forces obeys the well-known equation

$$\frac{\partial h}{\partial t} = -\mathrm{div}\left[\frac{h^3}{3\mu}\left(\mathrm{grad}\left(\sigma\Delta h + \frac{A'}{6\pi h^3}\right) + \rho\mathbf{g}\right)\right], \tag{1.1}$$

where h is the thickness of the film $|\nabla h| \ll 1$, μ is the dynamic viscosity, σ is the surface tension coefficient, ρ is the density, and A' is determined from the Hamaker constants [4].

If the interface ($|\mathrm{Ca}| \ll 1$, $\mathrm{Ca} = \mu v_0/\sigma$) moves at a low velocity v_0 in a round or flat capillary (Fig. 1), the interface shape at a fairly long distance from the wall $h \gg h_0$ is close to its shape at equilibrium:

$$\sigma(R_1^{-1} + R_2^{-1}) = \rho g(x - x_0) + \sigma R^{-1} + \mathrm{const} \tag{1.2}$$

(the interface is symmetric about the capillary axis, R_1 and R_2 are the main curvature radii, and the x axis is in opposition to **g**, $g = |\mathbf{g}|$. In the plane of x and h, $R_1 \to R$ as $x \to x_0$; x_0 is the point of intersection of the interface (1.2) with the solid wall ($h = 0$ at $x = x_0$), $\dot{x}_0 = v_0$. The dynamics of a thin p-film does not depend on the second radius of curvature R_2 if the round capillary is uniform along its axis. Note that if the dynamics of the film is governed by capillary forces, the role of R_2 can be significant for a film of great length.

We consider the parameter h_0 that divides the region of small thicknesses ($h \lesssim h_0$), in which spreading flow is substantial, and the region of great thicknesses ($h \gg h_0$), in which (1.2) is valid. If $\alpha_0 \to 0$, then $h \sim h_0$,

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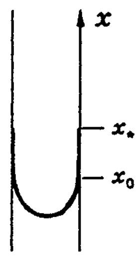

Fig. 1

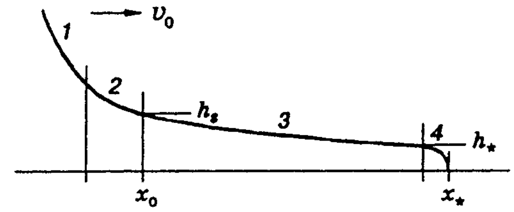

Fig. 2

which corresponds to $|x - x_0| \sim \delta \sim \sqrt{h_0 R}$ because of the boundedness of the Bond number $\mathrm{Bo} = \rho g R^2/\sigma$ ($\mathrm{Bo} \lesssim 1$). From (1.2) follows the asymptotic ($\delta/R \to 0$) boundary condition for (1.1):

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{R} + \frac{\rho g}{\sigma}(x - x_0) + \ldots \qquad \text{at} \quad \frac{x - x_0}{\delta} \to -\infty.$$

Ignoring small values of $\sim \mathrm{Bo}\,\delta/R$, we write

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{R} + \ldots \qquad \text{at} \quad \frac{x - x_0}{\delta} \to -\infty, \quad |x - x_0| \ll R. \tag{1.3}$$

To this corresponds the parabolic profile of the meniscus:

$$h = (1/2R)(x - x_0)^2 - \alpha_0(x - x_0) + \ldots \tag{1.4}$$

(α_0 is the dynamic boundary angle). The parameter $h_0 \sim \alpha_0^2 R$ [1, 2].

To close the asymptotic formulation of the problem (1.1)–(1.4), it is sufficient to specify the fluid flow rate (which is variable in the general case) in the capillary, the conditions for (1.1) at the wetting line, and the initial conditions. We assume that at the initial moment $t = 0$ the wetting line $x = x_*$ is close to the edge of the meniscus ($x_* = x_0$).

2. Asymptotic Model of the Dynamics of Slow Wetting. We assume that the time from the start of the wetting process is large ($t \to \infty$) and that the meniscus velocity v_0 is fairly low and changes smoothly with time. In this case, the scale l ($l \sim h/|\nabla h|$) of variation in wetting-film thickness for $x > x_0$ is sufficiently large; and $l_0 \gg \delta$ in the limit $x \to x_0$ (but $x - x_0 \gg \delta$). In this case, we can ignore the contribution of the capillary forces to (1.1) and write the equation of dynamics of a p-film [3, 5]:

$$\frac{\partial h}{\partial t} = æ\Delta \ln h - \frac{\rho}{3\mu} g \nabla h^3, \quad \mathrm{v} = -\frac{æ}{h^2}\nabla h + \frac{\rho g}{3\mu} h^2, \quad æ = \frac{A'}{6\pi\mu}. \tag{2.1}$$

In a boundary layer near the meniscus $\sim \mathrm{Bo}\,\delta/R$, in which capillary forces are important, Eq. (1.1) can also be simplified. We introduce the following variables:

$$\bar{x} = (x - x_0)/\delta, \quad \bar{h} = h/h_s, \quad \bar{t} = t/\tau, \quad \delta = h_s^2/\lambda, \quad \tau_s = 3\mu\delta^4/(\sigma h_s^3), \quad \lambda = \sqrt{A'/(2\pi\sigma)} \tag{2.2}$$

(τ is the characteristic time of the process, and $h_0 = h_s$ for slow wetting). With accuracy up to small quantities $\sim \mathrm{Bo}\,\delta/R$, Eq. (1.1) in the boundary layer takes the form

$$\frac{\tau_s}{\tau}\frac{\partial \bar{h}}{\partial \bar{t}} - \frac{v_0 \tau_s}{\delta}\frac{\partial \bar{h}}{\partial \bar{x}} = -\frac{\partial}{\partial \bar{x}} \bar{h}^3 \frac{\partial^3 \bar{h}}{\partial \bar{x}^3} + \frac{\partial^2}{\partial \bar{x}^2} \ln \bar{h}. \tag{2.3}$$

The relaxation time τ_s (2.2) coincides with the relaxation time to a uniform state of a film region of length δ under the action of capillary forces [6]. Owing to the strong difference between the scale l_0 of flow of the p-film and the thickness of the boundary layer ($l_0 \gg \delta$), for Eq. (2.3) the following is true:

$$\bar{h} \to 1 \qquad \text{at} \quad \bar{x} \to \infty. \tag{2.4}$$

Large characteristic times correspond to slow wetting:

$$\tau \gg \tau_s, \quad \tau_v = \delta/|v_0| \gg \tau_s \tag{2.5}$$

($\tau_s \to 0$), when the left side of (2.3) is negligibly small in a boundary layer ($\bar{x} \sim 1$), the equation for which, taking into account (2.4), is written in the form

$$\sigma\frac{\partial^2 h}{\partial x^2} + \frac{A'}{6\pi h^3} = \frac{A'}{6\pi h_s^3}. \tag{2.6}$$

In the limit of (2.4), we have $h - h_s \sim h_s \exp(-\bar{x})$. From (1.3) and Eq. (2.6) we find

$$h_s = (\lambda^2 R/3)^{1/3}, \quad \delta = \sqrt{h_s R/3}, \quad \lambda^2 = A'/(2\pi\sigma). \tag{2.7}$$

To the limit (2.4) of the solution in the boundary layer corresponds the following boundary condition for Eq. (2.1):

$$h = h_s \quad \text{at} \quad x = x_0. \tag{2.8}$$

At the wetting line, the following conditions [3, 5] must hold:

$$h = h_*, \quad \frac{dx_*}{dt} = v \quad \text{at} \quad x = x_*. \tag{2.9}$$

For steady motion of the p-film, conditions (2.9) agree with the sheet truncation effect [7].

One can show that the scale of the solution of problem (2.1), (2.8), and (2.9) at $x = x_0$ is sufficiently large ($l_0 \gg \delta$), if (2.5) holds. This is true, in particular, for exact solutions considered below. The problem of spreading of a wetting film, obviously, makes sense for $h_s > h_*$, since $h_s = h_*$ corresponds to equilibrium.

Thus, for slow wetting, the solution of Eq. (1.1) has the structure shown in Fig. 2, where 1 is the meniscus, 2 is the boundary layer between the p-film and the meniscus, 3 is the p-film, and 4 is the boundary layer in the wetting line [3]. Owing to this, the dynamics of the wetting p-film is determined by solution of Eq. (2.1) subject to boundary conditions (2.8) and (2.9), in which the function $x_0(t)$ is known and $x_*(t)$ is to be determined. The new model differs from the well-known model of [3], which contains the condition of unlimited thickness $h = \infty$ for $x = x_0$.

We note that the boundedness of h at the point x_0, generally speaking, cancels the equality $v = v_0$ [3], which holds because in a small neighborhood of a singular point a stationary solution is valid. In addition, there is a new possibility of describing flows for $v_0 < 0$, when the interface departs from the capillary.

Conditions (2.8) and (2.9) for the one-dimensional problem can be extended to the two-dimensional case. For (2.9), this extension is given in [3].

Below, we shall study the intermediate asymptotes in the limit $t \to \infty$, for which the acceleration of gravity (2.1) ($g = 0$) can be ignored. It is interesting to find out whether the motion of fairly long wetting films (of length of the order of the capillary meniscus rise) can be described within the framework of such asymptotes.

3. Stationary Solutions. We consider solutions of the form $h(x - v_0 t)$, which are stationary relative to the meniscus. If a solid surface remote from the meniscus is covered with a superthin film of constant thickness h_+, the solution of (2.1) subject to condition (2.8) has the form

$$h = h_+\left[1 + \left(\frac{h_+}{h_s} - 1\right)\exp\left(-\frac{h_+}{æ}v_0(x - x_0)\right)\right]^{-1}, \quad \dot{x_0} = v_0. \tag{3.1}$$

For steady flow in the stationary system ($v_0 = 0$), we have thickness $h = h_s \exp(-Qx/æ)$, $Q = \text{const}$.

Note that conditions (2.9) are not fulfilled for (3.1) at any point $x \in (x_0(t), \infty)$. Therefore, there is no analogy between problems of the steady spreading of a film over dry and wet surfaces. Interestingly, such an analogy is possible for nonsteady spreading.

For $h_+ \to 0$, from (3.1) follows a film profile that is analogous to [1, 7, 8]:

$$h = \left(\frac{1}{h_s} + \frac{v_0}{æ}(x - x_0)\right)^{-1}. \tag{3.2}$$

It is important that, in contrast to (3.1), the conditions at the wetting line (2.9) can be fulfilled for (3.2).

For flow of a thin layer with a dynamic boundary angle [1], the parameter h_0, which is equal to h_s in the absence of the angle, exceeds considerably ($h_0 \gg h_m$) the characteristic maximum thickness of a p-film

$$h_m = \sqrt{A'/2\pi\sigma}(3\mathrm{Ca})^{-1/3}. \tag{3.3}$$

Note that for slow wetting the parameter h_s coincides with the maximum thickness of the film.

4. Criteria for Limiting Wetting Modes. Let us compare the model of a boundary layer in the neighborhood of a meniscus with the dynamic boundary angle model [1]. According to (2.2) and (2.7), the relaxation time of the boundary layer near the meniscus is

$$\tau_s = \mu R^2/(3\sigma h_s). \tag{4.1}$$

To the condition of slow wetting $\tau_v = \delta/|v_0| \gg \tau_s$, by virtue of (2.7), (3.3), and (4.1), the following relationship of the maximum thicknesses in different modes corresponds: $h_s \ll h_m$. Therefore, the meniscus emerges starting from characteristic thicknesses h_s that are considerably smaller than the parameter h_m, in contrast to the flow mode with a dynamic boundary angle ($h_0 \gg h_m$). Thus, h_m in (3.3) has the meaning of a parameter that demarcates the limiting flow modes.

Let us find the minimum angle α_* and the corresponding capillary number Ca_* which restrict the validity of the formula for the dynamic boundary angle of a meniscus [1, 2]

$$\alpha_0^3 = 9\mathrm{Ca}(\ln(h_0/h_m) - (1/3)\ln\ln(h_0/h_m)), \quad h_0 = (2/e^2)\alpha_0^2 R \tag{4.2}$$

(the refined value of h_0 corresponds to [9]). The angle α_0 exists for $h_0 \gg h_m$ and makes no sense for $\ln(h_0/h_m) \sim 1$. Also, from (2.7), (3.3), and (4.2) follows the relation $h_m \sim h_s$ or an equivalent relation $\tau_v \sim \tau_s$. Hence, we obtain the following restriction on (4.2):

$$\alpha_0 \gg \alpha_* = 2.4(\lambda/R)^{1/3} \quad \text{or} \quad \mathrm{Ca} \gg \mathrm{Ca}_* = \lambda/R. \tag{4.3}$$

The conditions $h_s \ll h_m$ and (4.3) can be combined by one dimensionless parameter $\Pi_* = \mathrm{Ca}R/\lambda$. The slow wetting mode is the case for $\Pi_* \ll 1$. The dynamic boundary angle is observed for $\Pi_* \gg 1$. Therefore, the criterion Π_* determines the asymptotic wetting modes in the simplest way.

5. Maximum Length of a p-Film for Different Modes. For the case where a dynamic boundary angle is present, by determining the lowest speed from the formula for the minimum capillary number (4.3), we find the maximum length of a p-film [3]

$$l = æ/(v_0 h_*) < 0.2R\lambda/h_* \ll R. \tag{5.1}$$

The last inequality is true because $h_* \gg \lambda \sim a$ (a is the molecule size). From (5.1) follows an important conclusion: in the presence of a dynamic boundary angle, the length of the p-film is always much smaller than the curvature radius of the meniscus.

We estimate the growth of the length of the p-film during the characteristic relaxation time τ_s of the meniscus and the boundary layer to equilibrium. Gravity does not influence this process, and hence, we can assume that $g = 0$.

We consider first quasisteady flow whose velocity v_0 changes relatively slowly, and the film length is small. Substitution of the solution of (3.2) into (2.1) yields a quasi-steadiness criterion [3]:

$$|dv_0^{-1}/dt| \ll 1/l. \tag{5.2}$$

The maximum value of l is achieved for $h_s \gg h_*$: $l \approx æ/(h_* v_0)$. Correspondingly, (5.2) takes the form

$$|dl^2/dt| \ll 2æ/h_*.$$

Taking into account the initial condition $l = 0$ for $t = 0$, as a result of integration, we obtain the following restriction on the film length:

$$l \ll \sqrt{2æt/h_*}. \tag{5.3}$$

Note that for $v_0 \sim t^n$ and $t \to \infty$, condition (5.2) is fulfilled if $n > -1/2$.

We estimate l using the unsteady theory. Assuming, for simplicity, that $h_s = \infty$ and $x_0 = \text{const}$ and following the solution in [3] of the problem of a stationary piston, we calculate the length

$$l = \sqrt{8æt/(eh_*)}, \tag{5.4}$$

which is significantly larger than the estimate (5.3). Substituting $t = \tau_s$ from (4.1) into (5.4), we find

$$l(\tau_s) \sim 0.6R\lambda/\sqrt{h_* h_s} \ll R, \tag{5.5}$$

which is true by virtue of $\lambda \ll h_*$, h_s. According to (5.5), the growth in the length of the wetting film during the time required for stabilization of the slow wetting model is negligibly small compared with the characteristic size of the meniscus R. Therefore, to determine the film length, which is not small compared with the meniscus radius R, it is sufficient to use only the slow wetting model, ignoring the effects of meniscus motion at relatively high speeds.

6. Dynamics of Wetting of a Thin Capillary. The above analysis of the characteristic times is sufficiently complete if the equilibrium height H of the capillary meniscus rise is comparable with its radius ($H \sim R$). If the height of meniscus rise is large ($H \gg R$), which is the case in a thin capillary, new characteristic length and time scales emerge.

For large values of the fluid column length x_0 ($x_0 \gg R$), the spreading motion in the capillary is nearly Poiseuille, and the equation of motion of the meniscus takes the form

$$p_i + \frac{\sigma\gamma}{R} - \frac{p_e^{(0)}}{1 - k_p x_0} - x_0 \rho g = \left(k_v + x_0 \frac{\theta\mu}{R^2}\right) w, \quad x_0 = 0, \quad t = 0, \tag{6.1}$$

where p_i and k_v are the pressure and the coefficient of resistance ahead of the capillary inlet, $w = \dot{x_0}$ is the flow rate; $\gamma = 1$, $\theta = 3$ and $\gamma = 2$, $\theta = 8$, for the flat and axisymmetric cases, respectively; $p_e^{(0)}$ is the gas pressure ahead of the meniscus at $x_0 = 0$; and k_p is a coefficient that takes into account the gas volume change.

We consider the filling of a capillary under the action of capillary forces for $p_i = p_e^{(0)}$, $k_p = 0$, and $k_v = 0$. The dynamics of the meniscus in a small neighborhood of the equilibrium height H is defined by the formulas

$$x_0 - H = H \exp\left(-1 - \frac{t}{\tau_0}\right), \quad \tau_0 = \frac{H\theta\mu}{R^2 \rho g}, \quad H = \frac{\gamma\sigma}{\rho g R}. \tag{6.2}$$

The ratio of relaxation times (6.2) and (4.1) plays an important role in the dynamics of the p-film:

$$\frac{\tau_0}{\tau_s} = 3\gamma\theta \left(\frac{\sigma}{\rho g}\right)^2 \frac{h_s}{R^5}. \tag{6.3}$$

According to (6.3), for fairly thin capillaries ($R \to 0$), the characteristic time of stopping of the meniscus exceeds significantly the relaxation time of the boundary layer ($\tau_0/\tau_s \to 0$). Therefore, for such capillaries the slow wetting model is applicable not only when $x_0 \to H$, but also when the meniscus is relatively remote from its equilibrium position ($|x_0 - H| \sim H$). For this, it is necessary that (2.5) be satisfied, which, by virtue of (4.1), (6.1), and (6.3) is true when

$$\frac{x_0}{R} \gg \frac{H_1}{R} = \frac{\gamma}{3\theta}\left(\frac{R}{h_s}\right)^{3/2} = \frac{R}{\lambda}\frac{\gamma}{\theta\sqrt{3}}. \tag{6.4}$$

Obviously, for a vertical capillary this is possible for $H_1 \ll H$. From (6.4) and (6.2), we obtain the condition of validity of the slow wetting model for most of the path traversed by the meniscus:

$$R^3 \ll \frac{3\theta}{\rho g}\sqrt{\frac{A'\sigma}{6\pi}}. \qquad (6.5)$$

For example, for $A' = 10^{-20}$ J, $\sigma = 0.05$ N/m, and $\rho = 10^3$ kg/m^3, it follows from (6.5) that $R \ll 2\cdot 10^{-3}$ cm.

For capillary impregnation, $x_0 \sim \sqrt{t}$. Simple estimates show that in this case the p-film length is rather small ($l \ll x_0$). If condition (6.4) and quasi-steadiness condition (5.1) are fulfilled, solution (3.2) is admissible, which describes a quasi-stationary p-film adjacent to the meniscus. If (6.4) does not hold, for the greatest part of the path traversed by the meniscus a dynamic boundary angle can be observed. Note that the propagation of an extended p-film is a much slower process than capillary impregnation, and differences from the stationary theory of p-films can occur, as a rule, in a small neighborhood of the stopping point of the meniscus.

Equation of Motion of Meniscus. Let us refine the relation between the velocity of translational motion of the meniscus v_0 ($v_0 = \dot{x}_0$) and the flow rate w. Using the law of conservation of mass we write

$$v_0(S - h_s L) + v\Big|_{x_0} h_s L = wS$$

where S and L are the area and perimeter of the cross section of the capillary, the velocity v is given by (2.1), and $h_s \ll R$. Hence we obtain the refined formula

$$v_0 = w\left(1 + \gamma\frac{h_s}{R}\right) - v\Big|_{x_0}\frac{\gamma h_s}{R}, \qquad (6.6)$$

which is useful for a low flow rate w, for which the influence of fluid flow in a thin film on meniscus motion is important. Calculation of the meniscus velocity, according to (6.6), can be performed after problem (2.1), (2.8), and (2.9) is solved for $v_0 = w$.

7. Analytical Solutions of the Self-Similar Problems of Film Spreading. Problem (2.1), (2.8), and (2.9) has self-similar solutions at $g = 0$:

$$h = h_* y(\xi), \quad \xi = x\sqrt{\frac{h_*}{2æt}}. \qquad (7.1)$$

The dimensionless thickness of the film $y(\xi)$ is found from the boundary-value problem

$$-\xi y' = (\ln y)'', \ \ \xi \in (\xi_0, \xi_*), \ \ y = 1, \ \ y' = -\xi_*, \ \ \xi = \xi_*, \ \ y = y_s, \ \ \xi = \xi_0, \ \ y_s = h_s/h_*. \qquad (7.2)$$

The self-similar coordinate ξ_* of the wetting line, obviously, defines its velocity v_*:

$$\sqrt{t}v_* = \xi_*\sqrt{æ/2h_*}.$$

We shall consider important problems of film spreading that differ from (7.2).

Problem of Spreading of a Semi-Infinite Film. At the initial moment $t = 0$, the film has the shape of a step with height h_0: $h = h_0$ for $x < 0$ and $h = 0$ for $x > 0$.

In (7.2), instead of the last condition, the following condition must be fulfilled:

$$y \to y_0, \quad \xi \to -\infty. \qquad (7.3)$$

The spreading of a fluid over a surface covered by a thin liquid layer can be modeled by replacing condition (7.2) at the wetting line by the condition

$$y \to y_+ > 0, \quad \xi \to +\infty. \qquad (7.4)$$

To the problem of discontinuity decay for the film thickness ($t = 0$: $h = h_0$ for $x < 0$ and $h = h_+ < h_0$ for $x > 0$) correspond conditions (7.3) and (7.4).

For integration of Eq. (7.2), we introduce the variables ψ and φ according to the formulas

$$\psi = y'^2/y^3, \quad \varphi = \xi y'/y. \tag{7.5}$$

Equation (7.2) and its integral [3] are written as

$$-\xi \frac{d\psi}{d\xi} = \varphi\psi + 2\varphi^2; \tag{7.6}$$

$$\frac{\varphi^2}{\psi} + \varphi + \ln\psi - C = 0. \tag{7.7}$$

From (7.5)–(7.7), we find equations that give a solution in parametric form $y(\psi)$, $\xi(\psi)$:

$$\frac{d\xi}{\xi} = \mp\left(\frac{1}{\sqrt{1-4(\ln\psi - C)/\psi}} \mp 1\right)\frac{d\psi}{2\psi(\ln\psi - C)}; \tag{7.8}$$

$$y = \frac{\psi}{4\xi^2}\left(1 \pm \sqrt{1 - \frac{4}{\psi}(\ln\psi - C)}\right)^2. \tag{7.9}$$

Instead of (7.8), we write the equation

$$\frac{dy}{y} = \pm\frac{d\psi}{\sqrt{\psi^2 - 4\psi(\ln\psi - C)}}. \tag{7.10}$$

The general solution of problem (7.2) is constructed for $\xi_* > 0$. The case of $\xi_* < 0$ is obtained by the substitution $\xi \to -\xi$.

We shall treat ξ_* as a parameter. Then, from (7.2) and (7.7), we obtain

$$C = \ln\xi_*^2, \quad \psi(\xi_*) = \psi_* = \xi_*^2. \tag{7.11}$$

If $\psi_* \geqslant 4/e$, from (7.9)–(7.11) follow

$$\ln y = \int\limits_{\psi_*}^{\psi} \frac{d\psi}{\sqrt{\psi^2 - 4\psi\ln(\psi/\psi_*)}}, \quad \xi = \frac{1}{2}\left(\frac{\psi}{y}\right)^{1/2}\left(1 + \sqrt{1 - \frac{4}{\psi}\ln\frac{\psi}{\psi_*}}\right). \tag{7.12}$$

If $\psi_* \leqslant 4/e$, there are values of ψ_m and y_m such that the signs in formulas (7.8)–(7.10) change:

$$\psi_* = \psi_m \exp(-\psi_m/4), \quad \psi_m \leqslant 4, \quad \ln y_m = \int\limits_{\psi_*}^{\psi_m} \frac{d\psi}{\sqrt{\psi^2 - 4\psi\ln(\psi/\psi_*)}}. \tag{7.13}$$

In the range of $y \leqslant y_m$ and $\psi \in [\psi_*, \psi_m]$, formulas (7.12) are valid. In the range of $y \geqslant y_m$ and $\psi \in (0, \psi_m)$ the solution has the form

$$\ln y = \ln y_m + \int\limits_{\psi}^{\psi_m} \frac{d\psi}{\sqrt{\psi^2 - 4\psi\ln(\psi/\psi_*)}}, \quad \xi = \frac{1}{2}\left(\frac{\psi}{y}\right)^{1/2}\left(1 - \sqrt{1 - \frac{4}{\psi}\ln\frac{\psi}{\psi_*}}\right). \tag{7.14}$$

Solution of the Problem of a Stationary Meniscus. In the case of $\xi_0 = 0$, from (7.2), (7.13), and (7.14), we find

$$\ln y_s = 2\int\limits_0^{s_m} \frac{ds}{\sqrt{1 - (s/s_m)\exp(s_m - s)}}, \quad s_m \in (0,1), \quad s = \ln(\psi/\xi_*^2), \quad \xi_*^2 = 4s_m\exp(-s_m). \tag{7.15}$$

Hence follows the function $\xi_*(y_s)$.

In the limit $y_s \to 1$, the coefficient ξ_* tends to zero according to the law

$$\xi_* = \sqrt{y_s - 1} + \ldots, \tag{7.16}$$

which corresponds to a low spreading velocity.

The asymptote $y_s \to \infty$ corresponds to $s_m \to 1$. In this limit,

$$\int_0^{s_m} \frac{ds}{\sqrt{1-(s/s_m)\exp(s_m - s)}} = \sqrt{2}\ln\frac{2}{1-s_m} - 0.44248 + \ldots, \quad \xi_* = \frac{2}{\sqrt{e}}\left(1 - \frac{1}{4}(1-s_m)^2 + \ldots\right), \tag{7.17}$$

and hence we found

$$\xi_* = \frac{2}{\sqrt{e}}\left(1 - \frac{0.5349}{y_s^{\sqrt{2}/2}} + \ldots\right). \tag{7.18}$$

According to (7.18), $\xi_* \to 2/\sqrt{e}$ as $y_s \to \infty$.

The limit expression of the velocity of the wetting line

$$\sqrt{t}v_* = \sqrt{2æ/(eh_*)} \tag{7.19}$$

corresponds to the universal relation [3]. Formula (7.19) is true for a significantly large radius of the meniscus R, which is related to h_s by (2.7).

Solution of the Problem of Spreading of a Semi-Infinite Film. From (7.13) and (7.14), under condition (7.3), we obtain

$$\ln y_0 = \int_0^{s_m} \frac{ds}{\sqrt{1-(s/s_m)\exp(s_m - s)}} + \int_{-\infty}^{s_m} \frac{ds}{\sqrt{1-(s/s_m)\exp(s_m - s)}} \tag{7.20}$$

where s, s_m are defined in (7.15). Formulas (7.15) and (7.20) relate the coefficient ξ_* of the wetting line velocity to the film thickness y_0 at infinity. For small thicknesses ($y_0 \sim 1$), the coefficient ξ_* depends on y_0 similarly to (7.16):

$$y_0 - 1 = \xi_*\sqrt{\pi/2} + \ldots \tag{7.21}$$

Large thickness values ($y_0 \gg 1$) correspond to the limit $s_m \to 1$, where the integrals grow indefinitely. Formulas (7.17) and the following limit expression are valid:

$$\int_{-\infty}^{0} \frac{ds}{\sqrt{1-\dfrac{s}{s_m}\exp(s_m - s)}} = 1.06616 + \ldots, \quad s_m \to 1. \tag{7.22}$$

From (7.17), (7.20), and (7.22), we find

$$\xi_* = \frac{2}{\sqrt{e}}\left(1 - \frac{1.1367}{y_0^{\sqrt{2}/2}} + \ldots\right). \tag{7.23}$$

Thus, as $y_0 \to \infty$, the coordinate ξ_* approaches the limiting value $\xi_* = 2/\sqrt{e}$ in the problem of a stationary piston [3]. This proves that spreading of a film of large thickness obeys the universal relation (7.19) [3].

The solution of the problem of a stationary meniscus (7.15) is a part of the solution of the problem of spreading of a semi-infinite film.

The relationship between ξ_* and y_0 is given in Fig. 3 (curve 1). For comparison, graphs of asymptotes (7.21) and (7.23) are shown by curves 2 and 3. One can see that asymptotes (7.23) agree well with the exact solution and the velocity parameter ξ_* approximates (7.23) sufficiently rapidly with increasing film thickness y_0.

Fluid Spreading Over a Surface Covered by a Film. The solutions of the problems of wetting of a dry surface can be extended beyond the wetting line if we admit formally values $h < h_*$. For $y < 1$, we find a solution from formulas (7.12). If $\psi \to 0$, then $\xi \to \infty$ and $y \to y_+$. The limiting thickness is given by the formula

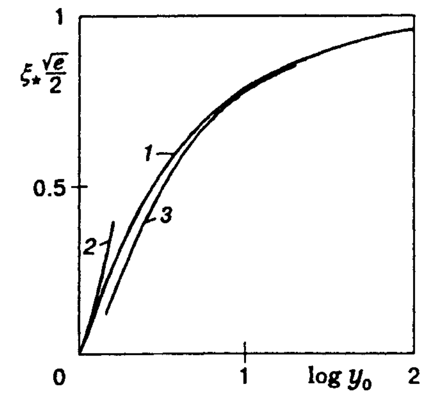

Fig. 3

$$\ln \frac{1}{y_+} = \int_0^\infty \frac{dz}{\sqrt{1 + 4\xi_*^{-2} z e^z}}. \tag{7.24}$$

Integral (7.24) equals the difference of the integrals in (7.20) for $\xi_* < 2/\sqrt{e}$ and coincides with integral (7.22) for $\xi_* = 2/\sqrt{e}$.

In the piston problem, the limiting thickness $y_+ \to 0$ as $\xi_0 \to \infty$. The reason for this is that $\xi_* \to \infty$ and integral (7.24) grows indefinitely. But small values $\xi_0 \to 0$ when $\xi_* \to 2/\sqrt{e}$ are of the greatest interest. In this case the above-mentioned integral is close to 1.

Thus, self-similar piston problems in the case where the surface is covered by a thin film with $\xi \to \infty$ or problems of thickness discontinuity decay are similar to problems of film spreading over a dry surface.

Another solution that corresponds to conditions $y \to y_0$ for $\xi \to -\infty$ and $y \to 0$ for $\xi \to \infty$ is interesting.

In this case, $y \to \xi^{-2}$ as $\xi \to \infty$ and $C = 2\ln 2 - 1$ in (7.8)–(7.10). The parameter $\psi = 4$ for $\xi = \infty$. The solution is given by (7.9) and by the formula

$$\ln \frac{y_0}{y} = \int_0^\psi \frac{d\psi}{\sqrt{\psi^2 - 4\psi(\ln\psi - C)}}.$$

The resulting solution describes the dynamics of a semi-infinite film in the range of rather large thicknesses, when the limiting thickness is large ($y_0 \to \infty$).

8. Range of Validity of the Intermediate Asymptotes for $t \to \infty$. Ignoring gravity in the dynamics of a wetting film is justified if the contribution of gravity to Eq. (2.1) is relatively small:

$$\rho g \ll \frac{A'}{2\pi h^4} \left|\frac{\partial h}{\partial x}\right| \geqslant \frac{A'}{2\pi h_s^4} \left|\frac{\partial h}{\partial x}(x_0)\right|. \tag{8.1}$$

The last relation is fulfilled for all exact solutions under consideration. This means that the contribution of the van der Waals forces to the dynamics of the film grows with a decrease in the film thickness. Among the self-similar solutions, the case of a stationary meniscus where the p-film length is maximal at any moment t is the most interesting. For this case, from formulas (7.1) and (7.5), we find

$$\left.\frac{\partial h}{\partial x}\right|_{x=0} = -\left.\sqrt{\psi}\right|_{\xi=0} \sqrt{\frac{h_s^3}{2\text{æ}t}} = -\xi_*^2 \frac{h_s}{l} \left(\frac{h_s}{h_*}\right)^{1/2}. \tag{8.2}$$

It is taken into account here that $\psi|_{\xi=0} = \psi_* = \xi_*^2$ by virtue of the second formula from (7.14). Considering large values of thickness ($h_s \gg h_*$) when $\xi_* = 2/\sqrt{e}$, we obtain from (8.1), by means of (2.7) and (8.2), the equivalent limitation of the p-film length:

$$l \ll 2\left(\frac{h_s}{h_*}\right)^{1/2} H, \quad H = \frac{\gamma\sigma}{\rho g R}. \tag{8.3}$$

Hence an important conclusion follows: the intermediate asymptotes for $t \to \infty$ ($g = 0$) are valid for wetting film lengths of the same order as the maximum height of meniscus rise ($l \sim H$). According to (8.3), a significant wetting effect due to van der Waals forces is achieved without a disturbing effect of g on film flow. The condition $h_s \gg h_*$ is fulfilled for not too small radii R of the meniscus. For example, for $h_* = 10^{-7}$ cm, $\sigma = 0.05$ N/m, and $A' = 10^{-20}$ J, it is necessary that $R \gg 10^{-5}$ cm. This is not a rigorous restriction, since $R = 10^{-5}$ cm corresponds to an enormous height of meniscus rise ($H = 100$ m) and a large right side of inequality (8.3).

The effect of the ratio h_s/h_* on the wetting rate for relatively large values of g can be studied using (2.1), (2.8), and (2.9). The lengths of p-films whose spreading is influenced by gravity are very large. Their propagation can be hindered, since the time of observation is limited.

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